

# Proof-theoretic strengths of weak theories for positive inductive definitions

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## Abstract

In this paper the lightface  $\Pi_1^1$ -Comprehension axiom is shown to be proof-theoretically strong even over  $\text{RCA}_0^*$ , and we calibrate the proof-theoretic ordinals of weak fragments of the theory  $\text{ID}_1$  of positive inductive definitions over natural numbers. Conjunctions of negative and positive formulas in the transfinite induction axiom of  $\text{ID}_1$  are shown to be weak, and disjunctions are strong. Thus we draw a boundary line between predicatively reducible and impredicative fragments of  $\text{ID}_1$ .

## 1 Introduction

This research is motivated to answer the questions raised in [7, 8]: Let  $|T|$  denote the proof-theoretic ordinal of a subsystem  $T$  of second order arithmetic.

**Conjecture 1.1** ([7, 8])

1.  $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \varphi\omega 0$ .
2.  $|\text{RCA}_0^* + (\Pi_1^1\text{-CA})^-| = \varphi\omega 0$ .
3.  $|\text{RCA}_0 + (\Pi_1^1\text{-CA})^-| = \vartheta(\Omega^\omega)$ .

where  $\text{RCA}_0^*$  is a subsystem of  $\text{RCA}_0$ , which is an extension of the elementary (recursive) arithmetic **EA** with induction axiom schema for bounded formulas.  $(\Pi_1^1(\Pi_3^0)\text{-CA})^-$  denotes the axiom schema of lightface, i.e., set parameter-free  $\Pi_1^1$ -Comprehension Axiom with  $\Pi_3^0$ -matrix  $\varphi$ :  $\exists Y \forall n [n \in Y \leftrightarrow \forall X \varphi(X, n)]$ .  $(\Pi_1^1\text{-CA})^-$  is the axiom schema of set parameter-free  $\Pi_1^1$ -Comprehension Axiom with arbitrary arithmetical formulas  $\varphi$ .

When  $\Sigma_1^0$ -formulas are available in the induction axiom schema, the proof-theoretic ordinal is shown to be the small Veblen ordinal  $\vartheta(\Omega^\omega)$  in [8].

**Theorem 1.2** ([8])  $|\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \vartheta(\Omega^\omega)$ .

According to [7] A. Weiermann showed that the wellfoundedness of ordinals up to each ordinal  $< \vartheta(\Omega^\omega)$  is provable in  $\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$ , and M. Rathjen showed that  $\text{RCA}_0 + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$  can be embedded in  $\Pi_2^1\text{-BI}_0$ , whose proof-theoretic ordinal  $|\Pi_2^1\text{-BI}_0| = \vartheta(\Omega^\omega)$ , cf. [6].

In trying to settle the Conjecture 1.1 affirmatively, we have first investigated weak fragments of the theory  $\text{ID}_1$  of positive inductive definitions over natural numbers, and found a line between predicatively reducible and impredicative fragments of  $\text{ID}_1$ , cf. Theorem 1.8. One fragment is proof-theoretically strong in the sense that the fragment proves the wellfoundedness up to each ordinal  $< \vartheta(\Omega^\omega)$ , cf. Lemma 2.2. The proof can be transformed to one in  $\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$ , and thereby we obtain  $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| \geq \vartheta(\Omega^\omega)$ . By combining Theorem 1.2 we arrive at a negative answer to the Conjecture 1.1.1,  $|\text{RCA}_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-| = \vartheta(\Omega^\omega)$ . Actually  $\text{RCA}_0^*$  is equal to  $\text{RCA}_0$  with the help of the lightface  $\Pi_1^1$ -Comprehension Axiom. Thus the whole of the Conjecture 1.1 is refuted.

Let  $\text{EA}^2$  be the elementary recursive arithmetic in the second order logic.  $\text{IND}$  denotes the  $\Pi_1^1$ -sentence  $\forall X \forall a [X(0) \wedge \forall y (X(y) \rightarrow X(y+1)) \rightarrow X(a)]$ .  $\Pi_k(\Omega)\text{-ID}(\Pi_1)$  is a fragment of  $\text{ID}_1$  defined in the next subsection. For a first-order positive formula  $\varphi(X, y)$ , let

$$I_\varphi = \bigcap \{X : \varphi(X) \subset X\}.$$

**Proposition 1.3** *Let  $k \geq 1$ .*

1.  $\text{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^- \vdash \Sigma_k^0\text{-IND}$ .
2. *Let  $\varphi(X, y)$  be a  $\Pi_1$ -positive formula. If  $\Pi_k(\Omega)\text{-ID}(\Pi_1) \vdash A(R_\varphi)$ , then  $\text{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^- \vdash A(I_\varphi)$ .*
3. *For any positive formula  $\varphi(X, y)$ , if  $\text{ID}_1 \vdash A(R_\varphi)$ , then  $\text{EA}^2 + \text{IND} + (\Pi_1^1\text{-CA})^- \vdash A(I_\varphi)$ .*

**Proof.** 1.3.1. Let  $k \geq 1$ . For a  $\Sigma_k^0$ -formula  $\varphi(a, X, z)$  let

$$N(a, z) := \forall X [\varphi(0, X, z) \wedge \forall y (\varphi(y, X, z) \rightarrow \varphi(y+1, X, z)) \rightarrow \varphi(a, X, z)]$$

$N(a, z)$  is a  $\Pi_1^1(\Sigma_{k+1}^0)$ -formula without set parameter, and is a set by  $(\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$ . It is obvious that  $N(a, z)$  is inductive with respect to  $a$ , i.e.,  $N(0, z)$  and  $\forall a [N(a, z) \rightarrow N(a+1, z)]$ . Therefore by  $\text{IND}$  we obtain  $\forall a N(a, z)$ , i.e.,  $\forall X [\varphi(0, X, z) \wedge \forall y (\varphi(y, X, z) \rightarrow \varphi(y+1, X, z)) \rightarrow \forall a \varphi(a, X, z)]$ .

1.3.2 and 1.3.3. We show Proposition 1.3.2. Proposition 1.3.3 is similarly seen. Argue in  $\text{EA}^2 + \text{IND} + (\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$ , and let  $\varphi$  be a positive  $\Pi_1$ -formula. First note that  $I_\varphi$  exists as a set by  $(\Pi_1^1(\Sigma_2^0)\text{-CA})^-$ . By Proposition 1.3.1 we have  $\Sigma_k^0\text{-IND}$ , and hence  $A(I_\varphi, 0) \wedge \forall x (A(I_\varphi, x) \rightarrow A(I_\varphi, x+1)) \rightarrow \forall z A(I_\varphi, z)$  for any  $\Sigma_k$ - or  $\Pi_k$ -formula  $A$ .  $\varphi(I_\varphi) \subset I_\varphi$  is seen logically.

Let  $A(I_\varphi, y)$  be a  $\Pi_k$ -formula. We show

$$\varphi(A(I_\varphi)) \subset A(I_\varphi) \rightarrow I_\varphi \subset A(I_\varphi) \tag{1}$$

Let  $B(y) :\Leftrightarrow \forall X [\varphi(A(X)) \subset A(X) \rightarrow A(X, y)]$ .  $B$  exists as a set by  $(\Pi_1^1(\Sigma_{k+1}^0)\text{-CA})^-$ . We claim that

$$\varphi(B) \subset B \quad (2)$$

Assume  $\varphi(B, y)$  and  $\varphi(A(X)) \subset A(X)$ . We need to show  $A(X, y)$ . We first show  $B \subset A(X)$ . Suppose  $B(z)$ . Then by the assumption  $\varphi(A(X)) \subset A(X)$  we have  $A(X, z)$ . Hence  $B \subset A(X)$ , and  $\varphi(B, y) \rightarrow \varphi(A(X), y)$  by the positivity of  $\varphi(X)$ . The assumption  $\varphi(B, y)$  yields  $\varphi(A(X), y)$ , and we conclude  $A(X, y)$  by  $\varphi(A(X)) \subset A(X)$ .

Since  $B$  is a set, we obtain  $I_\varphi \subset B$  by (2). On the other hand we have  $B(y) \rightarrow \varphi(A(I_\varphi)) \subset A(I_\varphi) \rightarrow A(I_\varphi, y)$  since  $I_\varphi$  is a set. Therefore  $\varphi(A(I_\varphi)) \subset A(I_\varphi) \rightarrow B \subset A(I_\varphi)$ . This together with  $I_\varphi \subset B$  yields (1).  $\square$

Let  $\Omega(1, \omega) = \Omega^\omega$ , and  $\Omega(n+1, \omega) = \Omega^{\Omega(n, \omega)}$  for  $n \geq 1$ .

**Theorem 1.4** *Let  $k \geq 1$ .*

$$1. \text{RCA}_0^* + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^- = \text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-.$$

$$2. \Pi_k(\Omega)\text{-ID}(\Pi_1) \text{ is interpreted canonically in } \text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-.$$

$$\begin{aligned} |\Pi_k(\Omega)\text{-ID}(\Pi_1)| &= |\text{RCA}_0^* + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-| \\ &= |\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-| = |\Pi_{k+1}^1\text{-BI}_0| = \vartheta(\Omega(k, \omega)) \end{aligned}$$

$$3. \text{ID}_1 \text{ is interpreted canonically in } \text{RCA}_0 + (\Pi_1^1\text{-CA})^-.$$

$$|\text{ID}_1| = |\text{RCA}_0^* + (\Pi_1^1\text{-CA})^-| = |\text{RCA}_0 + (\Pi_1^1\text{-CA})^-| = \vartheta(\epsilon_{\Omega+1}).$$

**Proof.** Let us consider Theorem 1.4.2. As in [8] using [4] we see that  $\Pi_{k+1}^1\text{-BI}_0$  comprises  $\text{RCA}_0 + (\Pi_1^1(\Pi_{k+2}^0)\text{-CA})^-$ . In [6] it is shown that  $|\Pi_{k+1}^1\text{-BI}_0| = \vartheta(\Omega(k, \omega))$ . Also it is well known that  $\vartheta(\Omega(k, \omega)) \leq |\Pi_k(\Omega)\text{-ID}(\Pi_1)|$ . Cf. Proof of Lemma 2.2 below for the case  $k = 1$ . Use  $\Pi_1(\Omega)$ -formula  $\sigma_1$  in (9) instead of  $\sigma_0$ .  $\square$

Let us introduce weak fragments of the theory  $\text{ID}_1$ .

## 1.1 Weak fragments

Let  $\mathcal{L}$  be a language for arithmetic having function constants<sup>1</sup> for each elementary recursive functions. Relation symbols in  $\mathcal{L}$  are  $=, <$ .  $\Delta_0^-$  denotes the set of bounded formulas in  $\mathcal{L}$ , and  $\Pi_0^{1-}$  the set of formulas in  $\mathcal{L}$ , called *arithmetical* formulas. The elementary recursive arithmetic  $\text{EA}$  is the theory in  $\mathcal{L}$  whose axioms are defining axioms for function constants, axioms for  $=, <$  and  $\Delta_0^-$ -IND: (3) is restricted to  $\theta \in \Delta_0^-$ .

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<sup>1</sup>The proof-theoretic strength does not increase with more constants, e.g., with function constants for primitive recursive functions.

For a second-order arithmetic  $T$ , its *proof-theoretic ordinal*  $|T|$  is defined to be the supremum of the order types  $|\prec|$  of elementary recursive and transitive relations  $\prec$  for which  $T \vdash \forall y(\forall x \prec y X(x) \rightarrow X(y)) \rightarrow \forall y X(y)$ . When  $T$  is a theory for positive inductive definitions,  $|T|$  is defined to be the supremum of the order types  $|\prec|$  of elementary recursive and transitive relations  $\prec$  for which  $T \vdash \forall x(x \in W_\prec)$  for the accessible (well founded) part  $W_\prec$  of the relation  $\prec$ .

For a class  $\Phi$  of  $X$ -positive formulas  $\varphi(X, x)$ , let  $\mathcal{L}(\Phi) = \mathcal{L} \cup \{R_\varphi : \varphi \in \Phi\}$  denote the language obtained from the language  $\mathcal{L}$  by adding unary predicate constants  $R_\varphi$  for each  $\varphi \in \Phi$ . The unary predicate constant  $R_\varphi$  is intended to denote the least fixed point of the monotone operator defined from  $\varphi$ :  $\mathbb{N} \supset \mathcal{X} \mapsto \{n \in \mathbb{N} : \mathbb{N} \models \varphi[\mathcal{X}, n]\}$ . Let us write  $\mathcal{L}(\text{ID})$  for  $\mathcal{L}(\Phi_{all})$ , where  $\Phi_{all}$  denotes the class of all  $X$ -positive formulas  $\varphi(X, x)$ .

The theory  $\text{ID}_1$  for non-iterated positive inductive definitions over natural numbers is an extension of the first-order arithmetic  $\text{PA}$ . Axioms are

$$1. \quad \theta(0) \wedge \forall x(\theta(x) \rightarrow \theta(x+1)) \rightarrow \theta(y) \quad (3)$$

for each  $\mathcal{L}(\text{ID})$ -formula  $\theta$ .

$$2. \quad \forall x[\varphi(R_\varphi, x) \rightarrow R_\varphi(x)] \quad (4)$$

$$3. \quad \forall u[R_\varphi(u) \rightarrow \forall x(\varphi(\sigma, x) \rightarrow \sigma(x)) \rightarrow \sigma(u)] \quad (5)$$

for each  $\mathcal{L}(\text{ID})$ -formula  $\sigma$ .

In weakening the theory  $\text{ID}_1$ , there are three factors. First, to which formulas  $\theta$  the complete induction schema (3) is available? Second, which class  $\Phi$  of positive formulas  $\varphi$  to define the least fixed point  $R_\varphi$  is to be considered? Third, which class  $\Gamma$  of formulas  $\sigma$  to which the axiom (5) is available?

For classes  $\Theta, \Gamma$  of formulas in  $\mathcal{L}(\Phi)$ , let  $(\Theta, \Gamma)\text{-ID}(\Phi)$  denote the fragment of  $\text{ID}_1$  defined as follows.  $(\Theta, \Gamma)\text{-ID}(\Phi)$  is an extension of  $\text{EA}$ . In  $(\Theta, \Gamma)\text{-ID}(\Phi)$ , the positive formula  $\varphi$  is in  $\Phi$ , the formulas  $\theta$  in complete induction schema (3) are in  $\Theta$ , and the formulas  $\sigma$  in the axiom (5) are in  $\Gamma$ .

When  $\Theta = \Gamma$ , let us write  $\Gamma\text{-ID}(\Phi)$  for  $(\Gamma, \Gamma)\text{-ID}(\Phi)$ , and when  $\Phi$  is the class  $\Phi_{all}$  of all positive formulas, let us write  $\Gamma\text{-ID}$  for  $\Gamma\text{-ID}(\Phi)$ .

$\text{Acc}$  denotes the class of  $X$ -positive formulas  $\varphi$

$$\varphi(X, x) \equiv [\forall y(\theta_0(x, y) \rightarrow t_0(x, y) \in X)] \quad (6)$$

with an arithmetic bounded formula  $\theta_0(x, y)$  and a term  $t_0(x, y)$ . In  $\theta_0(x, y)$  and  $t_0(x, y)$  parameters other than  $x, y$  may occur. For an elementary recursive relation  $\prec$ ,  $\forall y(y \prec x \rightarrow y \in X)$  is a typical example of an  $\text{Acc}$ -operator.  $\text{Acc}$  denotes the class of formulas  $\sigma(x)$  which are obtained from an  $\text{Acc}$ -operator by substituting any predicate constant  $R$  for  $X$

$$\sigma(x) \equiv [\forall y(\theta(x, y) \rightarrow t(x, y) \in R)] \quad (7)$$

where  $\theta(x, y)$  is an arithmetic bounded formula and  $t(x, y)$  a term possibly with parameters other than  $x, y$ .

**Definition 1.5** A formula is said to be *positive* [*negative*] if each predicate constant  $R_\varphi$  for least fixed point occurs only positively [negatively] in it, resp. Pos [Neg] denotes the class of all positive formulas [the class of all negative formulas], resp.

Also let  $P \cup N := \text{Pos} \cup \text{Neg}$ ,  $P \wedge N := \{C \wedge D : C \in \text{Pos}, D \in \text{Neg}\}$  and  $N \vee P := \{D \vee C : D \in \text{Neg}, C \in \text{Pos}\}$ .

**Definition 1.6** For  $k \geq 0$ , classes  $\Pi_k(P)$  and  $\Sigma_k(P)$  of formulas in the language  $\mathcal{L}(\text{ID})$  are defined recursively.

1.  $\Pi_0(P) = \Sigma_0(P)$  denotes the class of bounded formulas in positive formulas. Each formula in  $\Pi_0(P)$  is obtained from positive formulas by means of propositional connectives  $\neg, \vee, \wedge$  and bounded quantifiers  $\exists x < t, \forall x < t$ .
2.  $\Pi_k(P) \cup \Sigma_k(P) \subset \Pi_{k+1}(P) \cap \Sigma_{k+1}(P)$ .
3. Each class  $\Pi_k(P)$  and  $\Sigma_k(P)$  is closed under positive boolean combinations  $\vee, \wedge$  and bounded quantifications.
4. If  $A \in \Pi_k(P)$  [ $A \in \Sigma_k(P)$ ], then  $\neg A \in \Sigma_k(P)$  [ $\neg A \in \Pi_k(P)$ ], resp.
5. If  $A \in \Pi_k(P)$  [ $A \in \Sigma_k(P)$ ], then  $\forall x A \in \Pi_k(P)$  [ $\exists x A \in \Sigma_k(P)$ ], resp.

Let  $\Pi_\infty(P) = \bigcup_{k < \omega} \Pi_k(P)$ .

Classes  $\Pi_k(\Omega)$  and  $\Sigma_k(\Omega)$  are defined similarly by letting  $\Pi_0(\Omega) = \Sigma_0(\Omega)$  denote the class of bounded formulas in atomic formulas  $t \in R_\varphi, t = s, t < s$ . The predicates  $R_\varphi$  may occur positively and/or negatively in  $\Pi_0(\Omega)$ -formulas.

P-ID denotes the theory, in which  $X$ -positive formulas  $\varphi(X, x)$  are arbitrary, the formulas  $\theta$  in the the complete induction schema (3) as well as the formulas  $\sigma$  in the axiom (5) are restricted to positive formulas  $\theta, \sigma \in \text{Pos}$ .  $(\Pi_\infty(P), P)$ -ID is an extension of P-ID in which the formulas  $\theta$  in the the complete induction schema (3) are arbitrary, but the formulas  $\sigma$  in the axiom (5) are restricted to positive formulas  $\sigma \in \text{Pos}$ .

The following theorem is shown by D. Probst[5], and independently by B. Afshari and M. Rathjen[1]. P-ID  $[(\Pi_\infty(P), P)\text{-ID}]$  is denoted as  $\text{ID}_1^* \upharpoonright [\text{ID}_1^*]$  in [5], and as  $\text{ID}_1^* [\text{ID}_1^* + \text{IND}_{\mathbb{N}}]$  in [1], resp.

**Theorem 1.7** (Probst[5], Afshari and Rathjen[1])

1.  $|\text{P-ID}| = \varphi\omega 0 = \vartheta(\Omega \cdot \omega)$ .
2.  $|(\Pi_\infty(P), P)\text{-ID}| = \varphi\varepsilon_0 0 = \vartheta(\Omega \cdot \varepsilon_0)$ .

In this paper we show the following theorem 1.8.  $(\text{Acc}, N \vee P)\text{-ID}(\text{Acc})$  is the theory, in which  $X$ -positive formulas  $\varphi(X, x)$  are restricted to **Acc**-operators (6), the formulas  $\theta$  in the complete induction schema (3) are restricted to  $\theta \in \text{Acc}$  (7),

and the formulas  $\sigma$  in the axiom (5) are restricted to a disjunction of negative formula and a positive formula  $\sigma \in N \vee P$ .

$(\Pi_k(P), P \cup N)$ -ID denotes the theory, in which  $X$ -positive formulas  $\varphi(X, x)$  are arbitrary, the formulas  $\theta$  in the complete induction schema (3) are restricted to  $\theta \in \Pi_k(P)$ , and the formulas  $\sigma$  in the axiom (5) are restricted to a positive or negative formulas  $\sigma \in P \cup N$ .

$(\Pi_k(P), P \wedge N)$ -ID(Acc) is the theory, in which  $X$ -positive formulas  $\varphi(X, x)$  are restricted to Acc-operators (6), and the formulas  $\sigma$  in the axiom (5) are restricted to a conjunction of positive formula and a negative formula  $\sigma \in P \wedge N$ .

$P$ -ID  $\subset (\Pi_0(P), P \cup N)$ -ID and  $(\Pi_\infty(P), P)$ -ID  $\subset (\Pi_\infty(P), P \cup N)$ -ID are obvious. Note that each  $(\Pi_k(P), P \cup N)$ -ID proves that  $R_\varphi$  is the fixed point of positive  $\varphi$ , i.e.,  $(\Pi_k(P), P \cup N)$ -ID  $\vdash R_\varphi = \varphi(R_\varphi)$  since  $\varphi(\varphi(R_\varphi)) \subset \varphi(R_\varphi)$  for the positive formula  $\varphi(R_\varphi, x)$ .

Let  $\omega_0 = 1$  and  $\omega_{n+1} = \omega^{\omega_n}$ .

**Theorem 1.8**    1.

$$|(Acc, N \vee P)\text{-ID}(Acc)| = |(Acc, \Pi_0(P))\text{-ID}(Acc)| = |\Pi_1(P)\text{-ID}(Acc)| = \vartheta(\Omega^\omega)$$

2.

$$\begin{aligned} |Acc\text{-ID}(Acc)| &= |(\Pi_0(P), P \cup N)\text{-ID}| \\ &= |(\Pi_0(P), P \wedge N)\text{-ID}(Acc)| = \varphi\omega_0 \end{aligned}$$

3. For each  $k > 0$

$$\begin{aligned} |(\Pi_k(P), Acc)\text{-ID}(Acc)| &= |(\Pi_k(P), P \cup N)\text{-ID}| \\ &= |(\Pi_k(P), P \wedge N)\text{-ID}(Acc)| = \varphi\omega_{1+k}0 \end{aligned}$$

In particular  $|(\Pi_\infty(P), Acc)\text{-ID}(Acc)| = |(\Pi_\infty(P), P)\text{-ID}| = |(\Pi_\infty(P), P \cup N)\text{-ID}| = |(\Pi_\infty(P), P \wedge N)\text{-ID}(Acc)| = \varphi\varepsilon_0$ .

Among other things this means that negative formulas  $\sigma$  in the axiom (5) does not raise the proof-theoretic ordinals.

Let us mention the contents of the paper. In Section 2 the easy halves in Theorem 1.8 are shown by giving some wellfoundedness proofs. In Section 3 theories to be considered are reformulated in one-sided sequent calculi. In Section 4 finitary proofs in sequent calculi are first embedded to infinitary derivations to eliminate cut inferences partially. The first step is needed to unfold complex induction formulas. Second finitary proofs and infinitary derivations are embedded to the operator controlled derivations due to W. Buchholz[2]. In the latter derivations, cut formulas are restricted to boolean combinations of positive formulas. The upperbounds of the proof-theoretic ordinals are obtained through collapsing and bounding lemmas.

## 2 Wellfoundedness proofs

First let us recall the notation system  $OT'(\vartheta)$  in [8].  $OT'(\vartheta)$  denotes a notation system of ordinals based on  $\{0, \Omega, +, \vartheta\}$ . We need only ordinal terms up to  $\vartheta(\Omega^\omega)$ , the small Veblen number.

1.  $0 \in OT'(\vartheta)$ . 0 is the least element in  $OT'(\vartheta)$ , and  $K(0) = \emptyset$ .
2. If  $0 \notin \{\beta_k : k < n\} \subset OT'(\vartheta) \cap \Omega$  with  $\omega > i_{n-1} > \dots > i_0$  ( $n > 0$ ) and  $i_0 > 0 \vee n > 1$ , then  $\Omega^{i_{n-1}}\beta_{n-1} + \dots + \Omega^{i_0}\beta_0 \in OT'(\vartheta)$ .  $K(\Omega^{i_{n-1}}\beta_{n-1} + \dots + \Omega^{i_0}\beta_0) = \{\beta_k : k < n\}$ .
3. If  $\beta \in OT'(\vartheta)$ , then  $\vartheta(\beta) \in OT'(\vartheta) \cap \Omega$ .  $K(\vartheta(\beta)) = \{\vartheta(\beta)\}$ .
4.  $\vartheta(\alpha) < \vartheta(\beta) \Leftrightarrow [\alpha < \beta \wedge K(\alpha) < \vartheta(\beta)] \vee [\vartheta(\alpha) \leq K(\beta)]$ .
5. Each ordinal  $\vartheta(\alpha)$  is defined to be additively closed. This means that  $\beta, \gamma < \vartheta(\alpha) \Rightarrow \beta + \gamma < \vartheta(\alpha)$ .

Note that the system  $OT'(\vartheta)$  is  $\omega$ -exponential-free except  $\vartheta(\alpha) = \omega^{\alpha_0}$  for some  $\alpha_0$ . An inspection of the proof in [8] shows that  $Acc\text{-}ID(Acc)$  suffices to prove the wellfoundedness of ordinals up to each ordinal  $< \vartheta(\Omega \cdot \omega)$ .

Let  $<$  be the elementary recursive relation obtained from the relation  $<$  on  $OT'(\vartheta)$  through a suitable encoding. For the formula  $\forall y(y < x \rightarrow y \in X)$  in  $Acc$ , let  $W$  denote the accessible part of  $<$ , and  $Prog(X) :\Leftrightarrow \forall \alpha[\forall \beta < \alpha(\beta \in X) \rightarrow \alpha \in X]$ . Then the axiom (4) states  $Prog(W)$ , and the axiom (5) runs  $\forall x[x \in W \rightarrow Prog(\sigma) \rightarrow \sigma(x)]$  for  $\sigma \in Acc$ .

**Lemma 2.1**  $Acc\text{-}ID(Acc) \vdash \forall \beta < \vartheta(\Omega \cdot k)(\beta \in W)$  for each  $k < \omega$ .

**Proof.** We see that the following are provable in  $Acc\text{-}ID(Acc)$ . Note that  $A, B, C \in Acc$  for the formulas  $A, B, C$  below.

- (a)  $x \in W \rightarrow \forall y(y < x \rightarrow y \in W)$  by the axiom (5) for the formula  $A(x) \Leftrightarrow \forall y(y < x \rightarrow y \in W)$ .
- (b)  $y \in W \rightarrow x \in W \rightarrow x + y \in W$  by the axiom (5) for the formula  $B(y) \Leftrightarrow (x + y \in W)$ .
- (c) Assume  $K(a) = \emptyset$ ,  $\forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W)$  and  $\zeta \in W \cap \Omega$ , where  $\alpha <_\Omega \beta :\Leftrightarrow (K(\alpha) \cup K(\beta) \subset W \wedge \alpha < \beta)$ . Then  $Prog(C)$  for  $C(\xi) :\Leftrightarrow (\xi < \zeta \rightarrow \vartheta(\Omega \cdot a + \xi) \in W)$ .

Suppose  $\xi < \zeta$  and  $\forall \eta < \xi C(\eta)$ . Then  $\xi \in W$  by  $\zeta \in W$ , and  $K(\xi) \subset W$ . We show  $\forall \alpha < \vartheta(\Omega \cdot a + \xi)(\alpha \in W)$  by  $Acc$ -induction on the length of  $\alpha$ . By (b) we can assume that  $\alpha = \vartheta(\beta)$ . If  $\alpha \leq K(\xi) \subset W$ , then  $\alpha \in W$ . Otherwise  $K(\beta) \subset W$  by IH, and  $\beta < \Omega \cdot a + \xi$ . We can assume  $\beta = \Omega \cdot a + \eta$  for an  $\eta < \xi$  by the assumption  $\forall \beta <_\Omega \Omega \cdot a(\vartheta(\beta) \in W)$ . We obtain  $\alpha \in W$  by  $C(\eta)$ .

(d)  $K(a) = \emptyset \rightarrow \forall \beta <_{\Omega} \Omega \cdot a (\vartheta(\beta) \in W) \rightarrow \forall \beta <_{\Omega} \Omega \cdot (a+1) (\vartheta(\beta) \in W)$ .

Assume  $K(a) = \emptyset$ ,  $\forall \beta <_{\Omega} \Omega \cdot a (\vartheta(\beta) \in W)$  and  $\beta <_{\Omega} \Omega \cdot (a+1)$ . We need to show  $\vartheta(\beta) \in W$ . We can assume  $\beta = \Omega \cdot a + \zeta$  for a  $\zeta < \Omega$ . By  $K(\beta) \subset W$  we have  $\zeta \in W$ . Then by (c) we have  $Prog(C)$ , which yields  $\forall \xi \in W \cap \zeta (\vartheta(\Omega \cdot a + \xi) \in W)$  by the axiom (5) for the *Acc*-formula  $C$ . From this we see that  $\forall \alpha < \vartheta(\Omega \cdot a + \zeta) (\alpha \in W)$  by *Acc*-induction on the length of  $\alpha$ , and hence  $\vartheta(\Omega \cdot a + \zeta) \in W$  as desired.

By (d) we obtain  $\forall \beta <_{\Omega} \Omega \cdot k C(\beta)$ , i.e.,  $\forall \beta <_{\Omega} \Omega \cdot k (\vartheta(\beta) \in W)$  for each  $k$ . Using this and (b), we see that  $\forall \beta < \vartheta(\Omega \cdot k) (\beta \in W)$  by *Acc*-induction on the length of  $\beta$ . This shows Lemma 2.1.  $\square$

The next lemma shows the power of disjunctions of negative and positive formulas, i.e., implications of positive formulas in the axiom (5). Note that our proof of the lemma is formalizable in  $RCA_0^* + (\Pi_1^1(\Pi_3^0)\text{-CA})^-$ .

**Lemma 2.2** (*Acc*,  $N \vee P$ )-ID(*Acc*)  $\vdash \forall \beta < \vartheta(\Omega^\ell) (\beta \in W)$  for each  $\ell < \omega$ .

**Proof.** Let  $<_{lx}$  denote the lexicographic ordering on  $OT'(\vartheta) \times OT'(\vartheta)$ , in which the first components are ordered in the ordering  $<$  on  $OT'(\vartheta)$  and the second components are ordered in the  $\omega$ -ordering  $<^{\mathbb{N}}$  on  $OT'(\vartheta) \subset \mathbb{N}$ :

$$(\xi, \gamma) <_{lx} (\zeta, \beta) :\Leftrightarrow (\xi < \zeta) \vee (\xi = \zeta \wedge \gamma <^{\mathbb{N}} \beta)$$

Let  $W_{lx}$  denote the accessible part of  $<_{lx}$ , which is the least fixed point of the operator  $\forall (\xi, \gamma) <_{lx} (\zeta, \beta) X(\xi, \gamma)$ . Let  $Prog_{lx}(X) :\Leftrightarrow \forall (\zeta, \beta) [\forall (\xi, \gamma) <_{lx} (\zeta, \beta) X(\xi, \gamma) \rightarrow X(\zeta, \beta)]$ .

$$\zeta \in W \rightarrow \forall \beta [(\zeta, \beta) \in W_{lx}] \quad (8)$$

This follows from  $Prog(D)$  for  $D(\zeta) \Leftrightarrow \forall \beta [(\zeta, \beta) \in W_{lx}]$  with the positive formula  $D \in \text{Pos} \subset N \vee P$ .  $Prog(D)$  is seen from *Acc*-induction on  $\beta$ .

Let for  $\beta \in M :\Leftrightarrow (K(\beta) \subset W)$  and  $C_0(\beta) :\Leftrightarrow (\vartheta(\beta) \in W)$ ,

$$\begin{aligned} \sigma_1(\zeta) &:\Leftrightarrow (M \cap (\alpha + \Omega^\ell \zeta) \subset C_0) \Leftrightarrow \forall \beta \sigma_0(\zeta, \beta) \\ \sigma_0(\zeta, \beta) &:\Leftrightarrow (\beta \in M \wedge \beta < \alpha + \Omega^\ell \zeta \rightarrow \vartheta(\beta) \in W) \end{aligned} \quad (9)$$

Note that the formula  $\sigma_1(\zeta) \in \Pi_1(P)$  and  $\sigma_0(\zeta, \beta) \in N \vee P$  since  $\beta \in M \Leftrightarrow (K(\beta) \subset W)$  is a positive formula.

Assuming  $G^\ell :\Leftrightarrow \forall \alpha (M \cap \alpha \subset C_0 \rightarrow M \cap (\alpha + \Omega^\ell) \subset C_0)$ , and  $M \cap \alpha \subset C_0$ , we show that

$$Prog_{lx}(\sigma_0)$$

Suppose  $\forall (\xi, \gamma) <_{lx} (\zeta, \beta) \sigma_0(\xi, \gamma)$ ,  $\beta \in M$  and  $\beta < \alpha + \Omega^\ell \zeta$ . We need to show  $\vartheta(\beta) \in W$ . We can assume that  $\beta = \alpha + \Omega^\ell \xi + \delta$  with  $\xi < \zeta$  and  $\delta < \Omega^\ell$ . We claim that  $M \cap \alpha_0 \subset C_0$  for  $\alpha_0 = \alpha + \Omega^\ell \xi$ . Let  $\gamma \in M \cap \alpha_0$ . We have  $(\xi, \gamma) <_{lx} (\zeta, \beta)$ ,  $\gamma \in M$  and  $\gamma < \alpha + \Omega^\ell \xi$ .  $\sigma_0(\xi, \gamma)$  yields  $\vartheta(\gamma) \in W$ , i.e.,  $\gamma \in C_0$ .  $G^\ell$  yields  $\beta \in M \cap (\alpha_0 + \Omega^\ell) \subset C_0$  from  $M \cap \alpha_0 \subset C_0$ . Thus  $\vartheta(\beta) \in W$ .



From  $Progl_x(\sigma_0)$  we obtain  $\forall(\zeta, \beta) \in W_{lx} \sigma_0(\zeta, \beta)$ . By (8) we conclude  $\forall \zeta \in W \forall \beta \sigma_0(\zeta, \beta)$ , which means  $\forall \zeta \in W \sigma_1(\zeta)$ .

We have shown  $G^\ell \rightarrow M \cap \alpha \subset C_0 \rightarrow W \subset \sigma_1$ , and hence we obtain  $G^\ell \rightarrow G^{\ell+1}$ . By meta induction on  $\ell$  we obtain  $(Acc, N \vee P)\text{-ID}(\text{Acc}) \vdash \forall \beta < \vartheta(\Omega^\ell)(\beta \in W)$ .  $\square$

Lemma 2.2 shows that

$$\vartheta(\Omega^\omega) \leq |(Acc, N \vee P)\text{-ID}(\text{Acc})| \leq |(Acc, \Pi_0(P))\text{-ID}(\text{Acc})| \leq |\Pi_1(P)\text{-ID}(\text{Acc})|.$$

The non-trivial halves of Theorem 1.8 follow from the following theorem. For a positive operator  $\varphi(X, x)$  and a number  $n$  in the least fixed point  $I_\varphi$  of the monotonic operator  $\omega \supset \mathcal{X} \mapsto \{n : \mathbb{N} \models \varphi[\mathcal{X}, n]\}$ ,  $|n|_\varphi := \min\{\alpha : n \in I_\varphi^{\alpha+1}\}$  denotes the inductive norm of  $n$ .  $Th(\mathbb{N})$  denotes the set of true arithmetic sentences.

**Theorem 2.3** 1. For each  $k \geq 0$  and positive operator  $\varphi(X, x)$ ,

$$Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID} \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega \cdot \omega_{1+k}).$$

2. For each  $k \geq 0$  and positive operator  $\varphi(X, x)$ ,

$$Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc}) \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega \cdot \omega_{1+k}).$$

3. For each  $\text{Acc}$ -operator  $\varphi(X, x)$ ,

$$Th(\mathbb{N}) + \Pi_1(P)\text{-ID}(\text{Acc}) \vdash R_\varphi(n) \Rightarrow |n|_\varphi < \vartheta(\Omega^\omega).$$

Our proof of Theorem 2.3 is based on an analysis of infinitary derivations through the operator controlled derivations due to W. Buchholz[2]. An ordinal notation system with the  $\psi$ -function (but without the exponential function below  $\Omega$ ) is convenient for our proof.

**Definition 2.4** Let  $\Omega$  be the least uncountable ordinal  $\omega_1$ , and  $\varepsilon_{\Omega+1}$  the next epsilon number above  $\Omega$ . Define simultaneously on ordinals  $\alpha < \varepsilon_{\Omega+1}$ , operators  $\mathcal{H}_\alpha$  on the power set of  $\varepsilon_{\Omega+1}$ , and ordinals  $\psi_\alpha$  as follows. Let  $X \subset \varepsilon_{\Omega+1}$ .

1.  $\{0, \Omega\} \cup X \subset \mathcal{H}_\alpha(X)$ .
2. If  $\Omega < \beta \in \mathcal{H}_\alpha(X)$ , then  $\omega^\beta \in \mathcal{H}_\alpha(X)$ .
3.  $\{\beta, \gamma\} \subset \mathcal{H}_\alpha(X) \Rightarrow \beta + \gamma \in \mathcal{H}_\alpha(X)$ .
4.  $\beta \in \mathcal{H}_\alpha(X) \cap \alpha \Rightarrow \psi\beta \in \mathcal{H}_\alpha(X)$ .

Let

$$\psi_\alpha := \min\{\beta \leq \Omega : \mathcal{H}_\alpha(\beta) \cap \Omega \subset \beta\}.$$

It is well known that  $\mathcal{H}_{\varepsilon_{\Omega+1}}(0)$  is a computable notation system, and  $\psi\alpha$  is in normal form if  $G\alpha < \alpha$  for  $\alpha \in \mathcal{H}_{\varepsilon_{\Omega+1}}(0)$ , where  $G0 = G\Omega = \emptyset$ ,  $G(\psi\alpha) = \{\alpha\} \cup G\alpha$ ,  $G\omega^\alpha = G\alpha$  and  $G(\beta + \gamma) = G\beta \cup G\gamma$ . Also it is shown the following in [3].

**Proposition 2.5**  $\vartheta(\Omega \cdot \omega_{1+k}) = \psi(\Omega^{\omega_{1+k}})$ ,  $\vartheta(\Omega \cdot \varepsilon_0) = \psi(\Omega^{\varepsilon_0})$  and  $\vartheta(\Omega^\omega) = \psi(\Omega^{\Omega^\omega}) = \psi(\omega^{\Omega^\omega})$ .

Let  $W$  denote the accessible part of  $<$  on  $\mathcal{H}_{\varepsilon_{\Omega+1}}(0)$ . The easy half of Theorem 1.8 follows from the following lemma.

**Lemma 2.6** For each  $\alpha < \psi(\Omega^{\omega_{1+k}})$ ,  $(\Pi_k(P), \text{Acc})\text{-ID}(\text{Acc}) \vdash \alpha \in W$ .

**Proof.** It is clear that  $\text{Acc-ID}(\text{Acc}) \subset (\Pi_0(P), \text{Acc})\text{-ID}(\text{Acc})$ , and we have (a) and (b) in the proof of Lemma 2.1 in hand. The following (e) and (f) are provable in  $\text{Acc-ID}(\text{Acc})$ .

- (e)  $G\beta < \beta \rightarrow [\forall \gamma < \beta(\mathbb{P}(\gamma) \subset W \rightarrow w(\gamma)) \leftrightarrow w(\beta)]$ , where  $w(\gamma) :\Leftrightarrow (G\gamma < \gamma \rightarrow \psi(\gamma) \in W)$  and  $\mathbb{P}(\gamma)$  denotes the set of ordinal terms  $\psi\alpha$  occurring in  $\gamma$ .

Assume  $G\beta < \beta$  and  $\forall \gamma < \beta(\mathbb{P}(\gamma) \subset W \rightarrow w(\gamma))$ . By *Acc*-induction on the length of  $\alpha$  we see that  $\forall \alpha < \psi\beta(\alpha \in W)$ . For  $\alpha = \psi\gamma$  with  $G\gamma < \gamma$ ,  $\mathbb{P}(\gamma) \subset W$  follows from IH and  $\mathbb{P}(\gamma) < \psi\gamma$ .

- (f)  $\text{Prog}(E)$  for  $E(a) :\Leftrightarrow (G(a) = \emptyset \rightarrow \forall \beta[w(\beta) \rightarrow w(\beta + \Omega^a)])$ .

It suffices to show  $E(a+1)$  assuming  $G(a) = \emptyset$  and  $E(a)$ , which follows from  $\text{Prog}(D)$ , and the axiom (5) for the *Acc*-formula  $D(\zeta) :\Leftrightarrow (\zeta < \Omega \rightarrow w(\beta + \Omega^a \zeta))$ .

From (f) we see that  $\text{Acc-ID}(\text{Acc}) \vdash \forall \beta < \Omega^n w(\beta)$ , i.e.,  $\text{Acc-ID}(\text{Acc}) \vdash \forall \alpha < \psi(\Omega^n)(\alpha \in W)$  for each  $n$ .

In what follows argue in  $(\Pi_k(P), \text{Acc})\text{-ID}(\text{Acc})$ . For a formula  $A$ , let  $j[A](\alpha) :\Leftrightarrow \forall \beta[\forall \gamma < \beta A(\gamma) \rightarrow \forall \gamma < \beta + \omega^\alpha A(\gamma)]$ . Then  $\text{Prog}(A) \rightarrow \text{Prog}(j[A])$  for  $A \in \Pi_k(P)$ .

Let  $E_1 = E$  for the formula  $E$  in (f), and  $E_{n+1} = j[E_n]$ .

Then  $E_n \in \Pi_n(P)$  and  $\text{Prog}(E_{k+1})$ . This yields  $E_{k+1}(n)$  for each  $n$ . Hence  $E_{k+1-m}(\omega_m(n))$  for each  $n$  and  $m \leq k$ , where  $\omega_0(n) = n$  and  $\omega_{m+1}(n) = \omega^{\omega_m(n)}$ , i.e.,  $\omega_m = \omega_m(1)$ . In particular  $E_1(\omega_k(n))$  for each  $n$ . Therefore  $w(\Omega^{\omega_k(n)})$  for each  $n$ . We conclude  $\forall \alpha < \psi(\Omega^{\omega_k(n)})(\alpha \in W)$  in  $(\Pi_k(P), \text{Acc})\text{-ID}(\text{Acc})$ .  $\square$

### 3 Sequent calculi for weak fragments

Let us reformulate  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID}$ ,  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc})$  and  $Th(\mathbb{N}) + \Pi_1(P)\text{-ID}(\text{Acc})$  in one-sided sequent calculi. We assume that for each predicate symbol  $R$ , its complement or negation  $\bar{R}$  is in the language. For example, we have negations  $\neq, \not<$  of the predicate constants  $=, <$ . Logical connectives are  $\vee, \wedge, \exists, \forall$ . Negations  $\neg A$  of formulas  $A$  are defined recursively by

de Morgan's law and elimination of double negations.  $A \rightarrow B$  denotes  $\neg A \vee B$  for formulas  $A, B$ .  $\neg A$  is also denoted by  $\bar{A}$ .

The followings are *initial sequents*.

1. (logical initial sequent)

$$\bar{L}, L, \Gamma \text{ where } L \text{ is a literal.}$$

2. (equality initial sequent)

$$t \neq s, \bar{L}(t), L(s), \Gamma \text{ for literals } L(x).$$

3. (arithmetical initial sequent)

$$A, \Gamma$$

where  $A$  is one of formulas  $t = t$ , a defining axiom for an elementary recursive function, and a true arithmetical sentence in  $\mathcal{L}$ .

Inference rules are  $(cut)$ ,  $(\exists)$ ,  $(\forall)$ ,  $(b\exists)$ ,  $(b\forall)$ ,  $(\vee)$ ,  $(\wedge)$ ,  $(R)$ ,  $(\bar{R})$ , and  $(ind)$ .

- 1.

$$\frac{\Gamma, \bar{C} \quad C, \Delta}{\Gamma, \Delta} (cut)$$

where  $C$  is the *cut formula* of the  $(cut)$ .

- 2.

$$\frac{A(t), \Gamma}{\Gamma} (\exists) \quad \frac{A(a), \Gamma}{\Gamma} (\forall) \\ \text{where } (\exists x A(x)) \in \Gamma \quad \text{where } a \text{ is an eigenvariable and } (\forall x A(x)) \in \Gamma.$$

- 3.

$$\frac{A(t), \Gamma \quad t < s, \Gamma}{\Gamma} (b\exists) \quad \frac{a \not< s, A(a), \Gamma}{\Gamma} (b\forall) \\ \text{where } (\exists x < s A(x)) \in \Gamma \quad \text{where } a \text{ is an eigenvariable and } (\forall x < s A(x)) \in \Gamma.$$

- 4.

$$\frac{A_i, \Gamma}{\Gamma} (\vee) \quad \frac{A_0, \Gamma \quad A_1, \Gamma}{\Gamma} (\wedge) \\ \text{for an } i = 0, 1 \text{ with } (A_0 \vee A_1) \in \Gamma \quad \text{where } (A_0 \wedge A_1) \in \Gamma.$$

5. For each theory the inference rule for the predicates  $R_\varphi$  is the following:

$$\frac{\varphi(R_\varphi, t), \Gamma}{\Gamma} (R)$$

with  $(R_\varphi(t)) \in \Gamma$ .

- (a) For the theory  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)$ -ID, the following  $(\bar{R})$  is the inference rule for  $\bar{R}_\varphi$ :

$$\frac{\bar{\varphi}(\sigma, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

with  $(\bar{R}_\varphi(t)) \in \Gamma$  and an eigenvariable  $a$ , where  $\varphi(X, x)$  is an  $X$ -positive operator, and  $\sigma \in P \cup N$ .

- (b) For the theory  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)$ -ID(Acc), let  $\sigma \equiv (\bar{D} \wedge C)$  for positive formulas  $D, C$ , and  $\varphi(X, x)$  an Acc-operator in (6). Then the following  $(\bar{R})$  is the inference rule for  $\bar{R}_\varphi$ :

$$\frac{\neg\varphi(\bar{D}, a) \vee \neg\varphi(C, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

with  $(\bar{R}_\varphi(t)) \in \Gamma$  and an eigenvariable  $a$ . Note that  $\varphi(\bar{D}, a) \wedge \varphi(C, a)$  is logically equivalent to  $\varphi(\sigma, a)$ .

- (c) For the theory  $Th(\mathbb{N}) + \Pi_1(P)$ -ID(Acc), let  $\sigma \equiv (\forall z \sigma_0(z))$  for  $\sigma_0 \in \Pi_0(P)$ , and  $\varphi(X, x)$  an Acc-operator  $\forall y \{\theta_0(x, y) \rightarrow t_0(x, y) \in X\}$  with an arithmetic bounded formula  $\theta_0(x, y)$  and a term  $t_0(x, y)$ . Let

$$\varphi_\sigma(x) := [\forall w \{\theta_0(x, p_0(w)) \rightarrow \sigma_0(p_1(t_1))\}]$$

for  $t_1 \equiv (t_0(x, p_0(w)))$  and inverses  $p_0, p_1$  of a surjective pairing function. Note that  $\varphi_\sigma(x) \leftrightarrow \varphi(\sigma, x)$  over EA. Then the following  $(\bar{R})$  is the inference rule for  $\bar{R}_\varphi$ :

$$\frac{\neg\varphi_\sigma(a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

with  $(\bar{R}_\varphi(t)) \in \Gamma$  and an eigenvariable  $a$ .

6.

$$\frac{\Delta, \theta(0) \quad \Delta, \bar{\theta}(a), \theta(a+1) \quad \bar{\theta}(t), \Delta}{\Delta} (ind)$$

where  $a$  is the eigenvariable.

- (a) The *induction formula*  $\theta \in \Pi_k(P)$  for  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)$ -ID and for  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)$ -ID(Acc).  
(b)  $\theta \in \Pi_1(P)$  for  $Th(\mathbb{N}) + \Pi_1(P)$ -ID(Acc).

Note that we can assume that when  $k = 0$ ,  $\theta \in \Pi_0(P)$  is either a formula  $\exists y < t \forall z < s \bigwedge_i (C_i \rightarrow D_i)$  for some positive formulas  $C_i, D_i$ , or its complement  $\forall y < t \exists z < s \bigvee_i (C_i \wedge \bar{D}_i)$ . When  $k > 0$ , we can assume that  $\theta \in \Pi_k(P)$  is of the form  $\forall x_k \exists x_{k-1} \cdots Q x_1 \theta_0$ , where  $Q = \forall$  if  $k$  is odd, and  $Q = \exists$  else, and  $\theta_0 \in \Pi_0(P)$  is one of formulas  $\exists y < t \forall z < s \bigwedge_i (C_i \rightarrow D_i)$  and  $\forall y < t \exists z < s \bigvee_i (C_i \wedge \bar{D}_i)$ .

A *proof* is defined from these initial sequents and inference rules.

## 4 Infinitary derivations

In what follows we assume that each formula has no free variable, and a closed term  $t$  is identified with the numeral  $n$  of the value of  $t$ . Furthermore assume that there occurs no bounded quantifiers in any formula. Each bounded quantifier  $\exists x < n B(x), \forall x < n B(x)$  is replaced by  $\bigvee_{i < n} B(i), \bigwedge_{i < n} B(i)$ , resp. In other words,  $\bigvee_{i < n} B(i), \bigwedge_{i < n} B(i)$  are formulas for formulas  $\{B_i\}_{i < n}$ .

### 4.1 $\omega$ -rule

Finitary proof in the sequent calculus for  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID}$  or for  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc})$  with  $k > 0$  is embedded in infinitary derivations with the  $\omega$ -rule:

$$\frac{\{\Gamma, A(n) : n \in \mathbb{N}\}}{\Gamma} \text{ with } (\forall x A) \in \Gamma.$$

Let  $I_\varphi^{<\Omega} := R_\varphi$  and  $\bar{I}_\varphi^{<\Omega} := \bar{R}_\varphi$ . A formula is said to be *positive* [*negative*] if the predicates  $\bar{I}_\varphi^{<\Omega}$  [the predicates  $I_\varphi^{<\Omega}$ ] does not occur in it.

Definition 1.6 is modified as follows.

- Definition 4.1** 1.  $\Pi_0(P) = \Sigma_0(P)$  denotes a class of formulas of the form  $\bigvee_i \bigwedge_j (C_{ij} \rightarrow D_{ij})$  for some positive formulas  $C_{ij}, D_{ij}$ , or its complement  $\bigwedge_i \bigvee_j (C_{ij} \wedge \bar{D}_{ij})$ .
2. If  $A \in \Sigma_k(P)$ , then  $(\forall x A) \in \Pi_{k+1}(P)$ . If  $A \in \Pi_k(P)$ , then  $(\exists x A) \in \Sigma_{k+1}(P)$ .

**Definition 4.2** The *degree*  $\text{dg}(A) < \omega$  of the formula  $A \in \bigcup_{k < \omega} (\Sigma_k(P) \cup \Pi_k(P))$  is defined as follows.

1.  $\text{dg}(A) = 0$  if no predicate  $I_\varphi^{<\Omega}, \bar{I}_\varphi^{<\Omega}$  occurs in  $A$ .
2.  $\text{dg}(A) = 1 + \min\{k : A \in \Sigma_k(P) \cup \Pi_k(P)\}$  if one of the predicates  $I_\varphi^{<\Omega}, \bar{I}_\varphi^{<\Omega}$  occurs in  $A$ .

**Definition 4.3** For finite sets  $\Gamma$  of formulas, ordinals  $a < \varepsilon_0$  and  $d < \omega$ ,

$$\vdash_d^a \Gamma$$

designates that there exists an infinitary derivation with its ordinal depth  $\leq a$  and its cut degree  $< d$ , where an infinitary derivation is a well founded tree of sequents locally correct with inference rules in the sequent calculus for  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID}$  or for  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc})$  except the inference rule  $(\forall)$  is replaced by the  $\omega$ -rule.

Let  $\Gamma[\vec{a}]$  be a sequent in the language of  $\mathcal{L}(\text{ID})$ , where  $\vec{a} = (a_1, \dots, a_p)$  is a list of free variables occurring in the sequent  $\Gamma[\vec{a}]$ . For lists  $\vec{n} = (n_1, \dots, n_p) \subset \mathbb{N}$  of natural numbers,  $\Gamma^*[\vec{n}] = \{A^*[\vec{n}] : A \in \Gamma\}$  and  $A^*[\vec{n}]$  denotes the result of replacing every occurrence of the variable  $a_i$  in the list  $\vec{a}$  by the natural number  $n_i$ ,  $R_\varphi$  by  $I_\varphi^{<\Omega}$ , and every occurrence of bounded quantifies  $\exists x < n B(x), \forall x < n B(x)$  by  $\bigvee_{i < n} B^*(i), \bigwedge_{i < n} B^*(i)$ , resp.

**Lemma 4.4** (Pre-embedding)

1. If  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID} \vdash \Gamma[\vec{a}]$  for  $k > 0$ , then there exists  $a < \omega_{1+k}$  such that  $\vdash_2^a \Gamma^*[\vec{n}]$  for any  $\vec{n}$ .
2. If  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc}) \vdash \Gamma[\vec{a}]$  for  $k > 0$ , then there exists  $a < \omega_{1+k}$  such that  $\vdash_2^a \Gamma^*[\vec{n}]$  for any  $\vec{n}$ .

**Proof.** Consider  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID}$ . Let  $P$  be a proof of the sequent  $\Gamma[\vec{a}]$ . By eliminating (*cut*)'s partially we may assume that any cut formula in  $P$  is either an arithmetical formula or an atomic formulas  $R_\varphi(t)$ . We see easily that there exists  $c < \omega^2$  such that  $\vdash_{2+k}^c \Gamma^*[\vec{n}]$  for any  $\vec{n}$  since  $\text{dg}(\theta) \leq 1 + k$  for the induction formula  $\theta \in \Sigma_k(P)$ . By cut-elimination we obtain  $\vdash_2^a \Gamma^*[\vec{n}]$  for  $a = 2_k(c) < \omega_{1+k}$  with  $2_0(c) = c$  and  $2_{m+1}(c) = 2^{2^m(c)}$ .

The lemma for  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc})$  is similarly seen.  $\square$

## 4.2 $\Omega$ -rule

The language  $\mathcal{L}^\infty(\text{ID})$  for the next infinitary calculus is obtained from the language  $\mathcal{L}(\text{ID})$  by deleting free variables, and adding unary predicate symbols  $I_\varphi^{<\alpha}, \bar{I}_\varphi^{<\alpha}$  for each positive operator  $\varphi$  and each  $\alpha < \vartheta(\Omega^\omega) = \psi(\omega^{\Omega^\omega})$ .

A formula in the language  $\mathcal{L}^\infty(\text{ID})$  is said to be *positive* [*negative*] if the predicates  $\bar{I}_\varphi^{<\Omega}$  [the predicates  $I_\varphi^{<\Omega}$ ] does not occur in it. In these formulas predicates  $I_\varphi^{<\alpha}, \bar{I}_\varphi^{<\alpha}$  may occur. Definition 4.1 of classes  $\Pi_k(P), \Sigma_k(P)$  of formulas and Definition 4.2 of the degree  $\text{dg}(A)$  of formulas  $A$  are modified according to this enlargement of positive/negative formulas. Specifically predicates  $I_\varphi^{<\alpha}, \bar{I}_\varphi^{<\alpha}$  may occur in formulas  $A$  with  $\text{dg}(A) = 0$ .

A closed term  $t$  is identified with the numeral  $n$  of the value of  $t$ .  $\Gamma, \Delta, \dots$  denote finite sets of formulas, *sequents*.

$I_\varphi^{<\alpha}$  is intended to denote the union  $\bigcup_{\beta < \alpha} I_\varphi^\beta$  of the  $\beta$ -th stage  $I_\varphi^\beta = \{n \in \mathbb{N} : \varphi(I_\varphi^{<\beta}, n)\}$  of the least fixed point  $I_\varphi^{<\Omega}$ , and  $\bar{I}_\varphi^{<\alpha}$  its complement. Thus for any ordinal  $\alpha$  and any natural number  $n$

$$I_\varphi^{<\alpha}(n) \leftrightarrow \exists \beta < \alpha \varphi(I_\varphi^{<\beta}, n)$$

For a sequent  $\Gamma$  of formulas,  $k(\Gamma)$  denotes the set of ordinals  $\alpha < \Omega$  such that one of predicates  $I_\varphi^{<\alpha}, \bar{I}_\varphi^{<\alpha}$  occurs in a formula in the set  $\Gamma$ .  $\mathcal{H}[\Theta](X) := \mathcal{H}(\Theta \cup X)$  for sets  $\Theta, X$  of ordinals and operators  $\mathcal{H} : X \mapsto \mathcal{H}(X)$  on the sets  $X$  of ordinals.

**Definition 4.5** *Inductive definition of  $\mathcal{H} \vdash_d^a \Gamma$ .*

Let  $\Gamma$  be a sequent,  $a < \Omega \cdot \varepsilon_0$  and  $d \leq 3$ .  $\mathcal{H} \vdash_d^a \Gamma$  holds if

$$\{a\} \cup k(\Gamma) \subset \mathcal{H} \tag{10}$$

and one of the followings holds:

**(initial)** There exists a true arithmetic formula  $A \in \mathcal{L}$  in  $\Gamma$ .

- ( $\vee$ ) There exist a formula  $(\bigvee_{i < n} A_i) \in \Gamma$  with  $n > 0$ ,  $a_0 < a$  and  $i < n$  such that

$$\mathcal{H} \vdash_d^{a_0} \Gamma, A_i$$

In what follows let us write it as an inference rule:

$$\frac{\mathcal{H} \vdash_d^{a_0} \Gamma, A_i}{\mathcal{H} \vdash_d^a \Gamma} (\vee)$$

- ( $\wedge$ ) There exist a formula  $(\bigwedge_{i < n} A_i) \in \Gamma$  and an  $a_0 < a$  such that

$$\frac{\{\mathcal{H} \vdash_d^{a_0} \Gamma, A_i\}_{i < n}}{\mathcal{H} \vdash_d^a \Gamma} (\wedge)$$

- ( $\exists$ ) There exist a formula  $(\exists x A(x)) \in \Gamma$ ,  $n \in \omega$  and  $a(n) < a$  such that

$$\frac{\mathcal{H} \vdash_d^{a(n)} \Gamma, A(n)}{\mathcal{H} \vdash_d^a \Gamma} (\exists)$$

- ( $\forall^\omega$ ) There exist a formula  $(\forall x A(x)) \in \Gamma$  and  $\{a(n)\}_{n \in \mathbb{N}}$  such that  $a(n) < a$  for any  $n$  and

$$\frac{\{\mathcal{H} \vdash_d^{a(n)} \Gamma, A(n)\}_n}{\mathcal{H} \vdash_d^a \Gamma} (\forall^\omega)$$

- ( $I^<$ ) There exist  $\alpha \leq \Omega$ ,  $(I_\varphi^{<\alpha}(n)) \in \Gamma$  and  $\beta < \alpha$  such that  $a(\beta) < a$  and

$$\text{if } X \text{ occurs in } \varphi(X, n), \text{ then } \beta < a \quad (11)$$

$$\frac{\mathcal{H}' \vdash_d^{a(\beta)} \Gamma, \varphi(I_\varphi^{<\beta}, n)}{\mathcal{H} \vdash_d^a \Gamma} (I^<)$$

where  $\mathcal{H}' = \mathcal{H}[\{\beta\}]$  if  $X$  occurs in  $\varphi(X, n)$ , and  $\mathcal{H}' = \mathcal{H}$  else.

- ( $\bar{I}^<$ ) There exist  $\alpha \leq \Omega$ ,  $(\bar{I}_\varphi^{<\alpha}(n)) \in \Gamma$  and  $\{a(\beta)\}_{\beta < \alpha}$  such that  $a(\beta) < a$  for any  $\beta < \alpha$  and

$$\frac{\{\mathcal{H}[\{\beta\}] \vdash_d^{a(\beta)} \Gamma, \neg \varphi(\bar{I}_\varphi^{<\beta}, n)\}_{\beta < \alpha}}{\mathcal{H} \vdash_d^a \Gamma} (\bar{I}^<)$$

- ( $Cl$ ) There exist a formula  $(n \in I_\varphi^{<\Omega}) \in \Gamma$  and  $a_0 < a$  such that

$$\frac{\mathcal{H} \vdash_d^{a_0} \Gamma, \varphi(I_\varphi^{<\Omega}, n)}{\mathcal{H} \vdash_d^a \Gamma} (Cl)$$

(cut) There exist a formula  $C$  and  $a_0 < a$  such that  $\text{dg}(C) < d$  and

$$\frac{\mathcal{H} \vdash_d^{a_0} \Gamma, \neg C \quad \mathcal{H} \vdash_d^{a_0} C, \Gamma}{\mathcal{H} \vdash_d^a \Gamma} \text{ (cut)}$$

**Lemma 4.6** (Embedding 1)

If  $Th(\mathbb{N}) + (\Pi_k(\mathbb{P}), \mathbb{P} \cup \mathbb{N})\text{-ID} \vdash \Gamma[\vec{a}]$ , there exist an  $a < \Omega \cdot \omega_{1+k}$  such that for any  $\vec{n} \subset \mathbb{N}$  and any operator  $\mathcal{H} = \mathcal{H}_\gamma$  with  $\gamma \geq 2$ ,  $\mathcal{H} \vdash_2^a \Gamma^*[\vec{n}]$ .

**Proof.** Note that  $1 = \psi 0, \omega = \psi 1 \in \mathcal{H}_2$ .

First consider the case  $k = 0$ . Pick a finitary proof of the sequent  $\Gamma[\vec{a}]$  in  $Th(\mathbb{N}) + (\Pi_0(\mathbb{P}), \mathbb{P} \cup \mathbb{N})\text{-ID}$ .

Logical initial sequents  $\Gamma, \bar{R}_\varphi(t), R_\varphi(t)$  turns to  $\mathcal{H} \vdash_0^\Omega \Gamma^*, \bar{I}_\varphi^{<\Omega}(n), I_\varphi^{<\Omega}(n)$ , which in turn follows from  $\mathcal{H}[\{\alpha\}] \vdash_0^{f(\alpha)} \bar{I}_\varphi^{<\alpha}(n), I_\varphi^{<\alpha}(n)$  for any  $\alpha < \Omega$  and any  $n \in \mathbb{N}$  with  $f(\alpha) = k\alpha$  for a  $k < \omega$ .

$$\frac{\Delta, \theta(0) \quad \Delta, \bar{\theta}(a), \theta(a+1) \quad \bar{\theta}(t), \Delta}{\Delta} \text{ (ind)}$$

We can assume that the bounded formula  $\theta(a)$  is of the form  $\exists x < t \forall y < s \bigwedge_{k < m} (\bar{C}_k \wedge D_k)$  for positive formulas  $C_k, D_k$ . Then  $\theta^*(i) \equiv \bigvee_{j < p} \bigwedge_{k < q} (\bar{C}_{ijk} \wedge D_{ijk})$  with  $\text{dg}(\theta^*(i)) = 1$ . The inference (ind) turns to a series of (cut)'s of cut formulas  $\theta^*(i)$  for an  $a < \Omega \cdot \omega$ .

$$\frac{\mathcal{H} \vdash_2^a \Delta^*, \theta^*(0) \quad \mathcal{H} \vdash_2^a \Delta^*, \bar{\theta}^*(0), \theta^*(1)}{\mathcal{H} \vdash_2^{a+1} \Delta^*, \theta^*(1)} \\ \vdots \\ \frac{\mathcal{H} \vdash_2^{a+n} \Delta^*, \theta^*(n) \quad \vdash_2^a \bar{\theta}^*(n), \Delta^*}{\mathcal{H} \vdash_2^{a+\omega} \Delta^*}$$

$$\frac{\varphi(R_\varphi, t), \Gamma}{\Gamma} \text{ (R)} \text{ turns to } \frac{\mathcal{H} \vdash_2^a \varphi(I_\varphi^{<\Omega}, n), \Gamma}{\mathcal{H} \vdash_2^{a+1} \Gamma} \text{ (Cl)}$$

Finally consider

$$\frac{\bar{\varphi}(\sigma, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} \text{ (}\bar{R}\text{)}$$

where  $\bar{R}_\varphi(t) \in \Gamma$  and  $\sigma \in \mathbb{P} \cup \mathbb{N}$ .

By IH we have a  $c < \Omega \cdot \omega_{1+k}$  such that  $\mathcal{H} \vdash_2^c \bar{\varphi}(\sigma^*, n), \sigma^*(n), \Gamma^*$  and  $\mathcal{H} \vdash_2^c \bar{\sigma}^*(n), \Gamma^*$ . We show by induction on  $\alpha < \Omega$  that for  $f(\alpha) = c + \omega\alpha + 1$  and any  $n \in \mathbb{N}$

$$\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{<\alpha}(n), \sigma^*(n) \tag{12}$$

By IH we have  $\mathcal{H}[\{\beta\}] \vdash_2^{f(\beta)} \Gamma^*, \bar{I}_\varphi^{<\beta}(n), \sigma^*(n)$  for any  $\beta < \alpha$  and  $n \in \mathbb{N}$ . From this we see that  $\mathcal{H}[\{\beta\}] \vdash_2^{f(\beta)+m} \Gamma^*, \bar{\varphi}(I^{<\beta}, n), \varphi(\sigma^*, n)$  for an  $m < \omega$ . ( $\bar{I}^{<}$ )



yields  $\mathcal{H}[\{\alpha\}] \vdash_2^{c+\omega\alpha} \Gamma^*, \bar{I}_\varphi^{\leq \alpha}(n), \varphi(\sigma^*, n)$ . A *(cut)* with  $\mathcal{H} \vdash_2^c \bar{\varphi}(\sigma^*, n), \sigma^*(n), \Gamma^*$  yields  $\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{\leq \alpha}(n), \sigma^*(n)$ . Here note that  $\varphi(\sigma^*, n) \in \mathbf{P} \cup \mathbf{N}$  and

$$\varphi(\sigma^*, n) \in \Pi_0(\mathbf{P}) \quad (13)$$

with  $\text{dg}(\varphi(\sigma^*, n)) = 1$ .

From (12) and  $(\bar{I}^<)$  we have  $\mathcal{H} \vdash_2^{c+\Omega} \Gamma^*, \bar{I}_\varphi^{\leq \Omega}(n), \sigma^*(n)$ . Finally a *(cut)* with  $\text{dg}(\sigma^*(n)) = 1$  yields  $\mathcal{H} \vdash_2^{c+\Omega+1} \Gamma^*$  for  $(\bar{I}_\varphi^{\leq \Omega}(n)) \in \Gamma^*$  and  $c + \Omega + 1 < \Omega \cdot \omega_{1+k}$ .

Next consider the case  $k > 0$ . By Lemma 4.4 there exists  $a < \omega_{1+k}$  such that  $\vdash_2^a \Gamma^*[\vec{n}]$  for any  $\vec{n}$ . We show by induction on  $a$  that

$$\vdash_2^a \Gamma \Rightarrow \exists c \leq \Omega \cdot (1 + a) (\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^c \Gamma)$$

Consider an  $\omega$ -rule. Let  $(\forall x A(x)) \in \Gamma$  and  $\vdash_2^{a(n)} \Gamma, A(n)$  with  $a(n) < a$  for any  $n$ . By IH we have  $\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^{c(n)} \Gamma, A(n)$  for  $c(n) \leq \Omega \cdot (1 + a(n))$ . Then by  $(\forall^\omega)$  we obtain  $\mathcal{H}[\mathbf{k}(\Gamma)] \vdash_2^c \Gamma$  for  $c = \sup_n \{\Omega \cdot (1 + a(n)) + 1\} \leq \Omega \cdot (1 + a)$ .  $\square$

**Lemma 4.7** (Embedding 2)

*If  $\text{Th}(\mathbb{N}) + (\Pi_k(\mathbf{P}), \mathbf{P} \wedge \mathbf{N})\text{-ID}(\text{Acc}) \vdash \Gamma[\vec{a}]$ , there exist an  $a < \Omega \cdot \omega_{1+k}$  such that for any  $\vec{n} \subset \mathbb{N}$  and any operator  $\mathcal{H} = \mathcal{H}_\gamma$  with  $\gamma \geq 2$ ,  $\mathcal{H} \vdash_2^a \Gamma^*[\vec{n}]$ .*

**Proof.** This is seen as in Lemma 4.6. Consider

$$\frac{\neg\varphi(\bar{D}, a) \vee \neg\varphi(C, a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

where  $\bar{R}_\varphi(t) \in \Gamma$  and  $\sigma \equiv (\bar{D} \wedge C)$  with positive formulas  $D, C$ . As in (12) we see for a  $c < \Omega \cdot \omega_{1+k}$  and  $f(\alpha) = c + \omega\alpha + 1$  that  $\mathcal{H}[\{\alpha\}] \vdash_2^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{\leq \alpha}(n), \sigma^*(n)$  for any  $n \in \mathbb{N}$ . Note that the cut formulas  $\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n)$  and  $\sigma^*(n)$  arise, cf. (13). We have  $\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n), \sigma^*(n) \in \Pi_0(\mathbf{P})$  and  $\text{dg}(\varphi(\bar{D}^*, n) \wedge \varphi(C^*, n)) = \text{dg}(\sigma^*(n)) = 1$ .  $\square$

**Lemma 4.8** (Embedding 3)

*If  $\text{Th}(\mathbb{N}) + \Pi_1(\mathbf{P})\text{-ID}(\text{Acc}) \vdash \Gamma[\vec{a}]$ , there exist an  $a < \Omega \cdot \omega$  such that for any  $\vec{n} \subset \mathbb{N}$  and any operator  $\mathcal{H} = \mathcal{H}_\gamma$  with  $\gamma \geq 2$ ,  $\mathcal{H} \vdash_3^a \Gamma^*[\vec{n}]$ .*

**Proof.** This is seen as in Lemma 4.6 for  $k = 0$ . Note that the cut formula  $\theta^*(i) \in \Pi_1(\mathbf{P})$  arises from *(ind)* with  $\text{dg}(\theta^*(i)) = 2$ . Consider

$$\frac{\neg\varphi_\sigma(a), \sigma(a), \Gamma \quad \bar{\sigma}(t), \Gamma}{\Gamma} (\bar{R})$$

where  $\bar{R}_\varphi(t) \in \Gamma$  and  $\sigma, \varphi_\sigma(a) \in \Pi_1(\mathbf{P})$ . As in (12) we see for a  $c < \Omega \cdot \omega$  and  $f(\alpha) = c + \omega\alpha + 1$  that  $\mathcal{H}[\{\alpha\}] \vdash_3^{f(\alpha)} \Gamma^*, \bar{I}_\varphi^{\leq \alpha}(n), \sigma^*(n)$  for any  $n \in \mathbb{N}$ . For the cut formulas  $\varphi_\sigma^*(n)$  and  $\sigma^*(n)$ , we have  $\varphi_\sigma^*(n), \sigma^*(n) \in \Pi_1(\mathbf{P})$  and  $\text{dg}(\varphi_\sigma^*(n)) = \text{dg}(\sigma^*(n)) = 2$ .  $\square$

**Lemma 4.9** *For any operator  $\mathcal{H} = \mathcal{H}_\gamma$  with  $\gamma \geq 2$ , if  $\mathcal{H} \vdash_3^a \Gamma$ , then  $\mathcal{H} \vdash_2^{\omega^a} \Gamma$ .*

In the following lemmas  $\Gamma^{(b)} = \{A^{(b)} : A \in \Gamma\}$ , while  $A^{(b)}$  is obtained from  $A$  by replacing some *positive* occurrences of  $I_\varphi^{<\Omega}$  by  $I_\varphi^{<b}$ .

**Lemma 4.10** (Bounding)

Let  $\mathcal{H} \vdash_1^a \Gamma$  for  $a < \Omega$  and  $\Gamma \subset \text{Pos}$ . Then  $\mathcal{H} \vdash_1^a \Gamma^{(b)}$  for  $a \leq b \in \mathcal{H} \cap \Omega$ .

**Proof.** This is seen by induction on  $a < \Omega$ .

Suppose  $\mathcal{H} \vdash_1^a \Gamma$  follows from  $(I^<)$  so that  $(I_\varphi^{<\Omega}(n)) \in \Gamma$  and  $\mathcal{H}[\{\gamma\}] \vdash_1^{a(\gamma)} \Gamma, \varphi(I_\varphi^{<\gamma}, n)$  with  $\gamma < \Omega$ ,  $a(\gamma) < a$  and  $\gamma < a$  if  $X$  occurs in  $\varphi(X, n)$ , (11). Then by IH we have  $\mathcal{H}[\{\gamma\}] \vdash_1^{a(\gamma)} \Gamma^{(b)}, \varphi(I_\varphi^{<\gamma}, n)$ . By  $(I^<)$  we obtain  $\mathcal{H} \vdash_1^a \Gamma^{(b)}$  for  $\gamma < a \leq b$ .

Suppose  $\mathcal{H} \vdash_1^a \Gamma$  follows from  $(Cl)$  so that  $(I_\varphi^{<\Omega}(n)) \in \Gamma$  and  $\mathcal{H} \vdash_1^{a_0} \Gamma, \varphi(I_\varphi^{<\Omega}, n)$  with  $a_0 < a$ . By IH we have  $\mathcal{H} \vdash_1^{a_0} \Gamma^{(b)}, \varphi(I_\varphi^{<a_0}, n)$  for  $a_0 < a \leq b$  and  $a_0 \in \mathcal{H}$ . An  $(I^<)$  yields  $\vdash_1^a \Gamma^{(b)}$ .  $\square$

**Lemma 4.11** (Collapsing)

Let  $\gamma \in \mathcal{H}_\gamma$  and  $\Gamma \subset \text{Pos}$ . Assume  $\mathcal{H}_\gamma \vdash_2^a \Gamma$ . Then  $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma$  for  $\hat{a} = \gamma + \omega^{\Omega+a}$ .

**Proof.** We show the lemma by induction on  $a$ .

First let us verify the condition (10) in  $\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma$ . From  $\gamma < \hat{a} + 1$  we see  $k(\Gamma) \subset \mathcal{H}_\gamma \subset \mathcal{H}_{\hat{a}+1}$ . Also by  $\{\gamma, a\} \subset \mathcal{H}_\gamma$  we have  $\hat{a} = \gamma + \omega^{\Omega+a} \in \mathcal{H}_\gamma \subset \mathcal{H}_{\hat{a}}$  and  $\psi\hat{a} \in \mathcal{H}_{\hat{a}+1}$ . From  $\hat{a} \in \mathcal{H}_{\hat{a}}$  we see that if  $a_0 < a$  and  $\mathcal{H}_\gamma \vdash_2^{a_0} \Gamma_0$ , then  $\psi\hat{a}_0 < \psi\hat{a}$ .

**Case 1.** For  $a(\beta) < a$

$$\frac{\{\mathcal{H}_\gamma[\{\beta\}] \vdash_2^{a(\beta)} \Gamma, \bar{I}_\varphi^{<\alpha}(n), \neg\varphi(I_\varphi^{<\beta}, n) : \beta < \alpha\}}{\mathcal{H}_\gamma \vdash_2^a \Gamma, \bar{I}_\varphi^{<\alpha}(n)} (\bar{I}^<)$$

From  $(\bar{I}_\varphi^{<\alpha}(n)) \in \text{Pos}$  we see  $\alpha < \Omega$ . We claim

$$\forall \beta < \alpha (\beta \in \mathcal{H}_\gamma) \quad (14)$$

Let  $\beta < \alpha$ . We have  $\Omega > \alpha \in k(\bar{I}_\varphi^{<\alpha}(n)) \subset \mathcal{H}_\gamma$ , which yields  $\beta < \alpha \in \mathcal{H}_\gamma(0) \cap \Omega = \psi\gamma$ , and  $\beta \in \mathcal{H}_\gamma$ . (14) yields  $\mathcal{H}_\gamma[\{\beta\}] = \mathcal{H}_\gamma$ .

By IH we obtain for  $\widehat{a(\beta)} = \gamma + \omega^{\Omega+a(\beta)}$ ,  $\widehat{\psi a(\beta)} < \psi\hat{a}$  and

$$\frac{\{\widehat{\mathcal{H}_{a(\beta)+1}} \vdash_1^{\psi\widehat{a(\beta)}} \Gamma, \bar{I}_\varphi^{<\alpha}(n), \neg\varphi(I_\varphi^{<\beta}, n) : \beta < \alpha\}}{\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma, \bar{I}_\varphi^{<\alpha}(n)} (\bar{I}^<)$$

**Case 2.** For  $\beta < \min\{\alpha, a\}$  and  $\alpha \leq \Omega$

$$\frac{\mathcal{H}_\gamma \vdash_2^{a(\beta)} \Gamma, I_\varphi^{<\alpha}(n), \varphi(I_\varphi^{<\beta}, n)}{\mathcal{H}_\gamma \vdash_2^a \Gamma, I_\varphi^{<\alpha}(n)} (I^<)$$

If  $X$  occurs in  $\varphi(X, n)$ , then by (10) we have  $\Omega > \beta \in \mathbf{k}(\varphi(I_\varphi^{<\beta}, n)) \subset \mathcal{H}_\gamma$ , and  $\beta < \psi\gamma \leq \psi\hat{a}$ . Hence (11) is enjoyed in the following inference. For  $\widehat{a(\beta)} = \gamma + \omega^{\Omega+a(\beta)}$  we obtain by IH

$$\frac{\mathcal{H}_{\widehat{a(\beta)+1}} \vdash_1^{\psi\widehat{a(\beta)}} \Gamma, I_\varphi^{<\alpha}(n), \varphi(I_\varphi^{<\beta}, n)}{\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi\hat{a}} \Gamma, I_\varphi^{<\alpha}(n)} (I^<)$$

**Case 3.**  $\mathcal{H}_\gamma \vdash_2^a \Gamma$  follows by a (*cut*) from

$$\mathcal{H}_\gamma \vdash_2^{a_0} \Gamma, \bigwedge_i (C_i \vee \bar{D}_i) \quad (15)$$

and

$$\mathcal{H}_\gamma \vdash_2^{a_0} \bigvee_i (\bar{C}_i \wedge D_i), \Gamma \quad (16)$$

for  $a_0 < a$  and positive formulas  $C_i, D_i$ .

By inversion on (16) we have

$$\mathcal{H}_\gamma \vdash_2^{a_0} \{D_i\}_i, \Gamma$$

By IH we obtain for  $\hat{a}_0 = \gamma + \omega^{\Omega+a_0}$  and  $\beta_1 = \psi\hat{a}_0$

$$\mathcal{H}_{\hat{a}_0+1} \vdash_1^{\beta_1} \{D_i\}_i, \Gamma$$

From  $\beta_1 \in \mathcal{H}_{\hat{a}_0+1}$  and the Bounding lemma 4.10 we obtain

$$\mathcal{H}_{\hat{a}_0+1} \vdash_1^{\beta_1} \{D_i^{(\beta_1)}\}_i, \Gamma \quad (17)$$

On the other hand we have by inversion on (15) for each  $i$

$$\mathcal{H}_{\hat{a}_0+1} \vdash_2^{a_0} \Gamma, C_i, \bar{D}_i^{(\beta_1)}$$

From  $\hat{a}_0 + 1 \in \mathcal{H}_{\hat{a}_0+1}$  and IH we obtain for  $\beta_2 = \psi(\gamma + \omega^{\Omega+a_0} \cdot 2)$

$$\mathcal{H}_{\gamma+\omega^{\Omega+a_0} \cdot 2+1} \vdash_1^{\beta_2} \Gamma, C_i, \bar{D}_i^{(\beta_1)}$$

Once again by the Bounding lemma 4.10 we obtain

$$\mathcal{H}_{\gamma+\omega^{\Omega+a_0} \cdot 2+1} \vdash_1^{\beta_2} \Gamma, C_i^{(\beta_2)}, \bar{D}_i^{(\beta_1)} \quad (18)$$

Again by inversion on (16) we have

$$\mathcal{H}_{\gamma+\omega^{\Omega+a_0} \cdot 2+1} \vdash_2^{a_0} \{\bar{C}_i^{(\beta_2)}\}_i, \Gamma$$

and IH yields for  $\beta_3 = \psi(\gamma + \omega^{\Omega+a_0} \cdot 3)$

$$\mathcal{H}_{\gamma+\omega^{\Omega+a_0} \cdot 3+1} \vdash_1^{\beta_3} \{\bar{C}_i^{(\beta_2)}\}_i, \Gamma \quad (19)$$

Several (*cut*)'s from (17), (18) and (19) yield

$$\frac{\mathcal{H}_{\gamma+\omega^{\Omega+a_0+1}} \vdash_1^{\beta_1} \{D_i^{(\beta_1)}\}_i, \Gamma \quad \{\mathcal{H}_{\gamma+\omega^{\Omega+a_0+2+1}} \vdash_1^{\beta_2} \Gamma, C_i^{(\beta_2)}, \bar{D}_i^{(\beta_1)}\}_i \quad \mathcal{H}_{\gamma+\omega^{\Omega+a_0+3+1}} \vdash_1^{\beta_3} \{\bar{C}_i^{(\beta_2)}\}_i, \Gamma}{\mathcal{H}_{\hat{a}+1} \vdash_1^{\psi \hat{a}} \Gamma}$$

Here we have  $\gamma + \omega^{\Omega+a_0} \cdot 3 < \gamma + \omega^{\Omega+a} = \hat{a}$  and  $\beta_1 = \psi(\gamma + \omega^{\Omega+a_0}) < \psi(\gamma + \omega^{\Omega+a_0} \cdot 2) = \beta_2 < \beta_3 = \psi(\gamma + \omega^{\Omega+a_0} \cdot 3) < \psi(\gamma + \omega^{\Omega+a}) = \psi \hat{a}$ .

All other cases are seen easily from IH.  $\square$

(**Proof** of Theorem 2.3). First consider Theorems 2.3.1 and 2.3.2. Let  $Th(\mathbb{N}) + (\Pi_k(P), P \cup N)\text{-ID} \vdash R_\varphi(n)$  for a positive operator  $\varphi(X, x)$ , or  $Th(\mathbb{N}) + (\Pi_k(P), P \wedge N)\text{-ID}(\text{Acc}) \vdash R_\varphi(n)$  for an **Acc**-operator  $\varphi(X, x)$ . By Embedding Lemmas 4.6 and 4.7, we have  $\mathcal{H}_2 \vdash_2^a I_\varphi^{<\Omega}(n)$  for  $0 < a < \Omega \cdot \omega_{1+k}$ . Then by Collapsing Lemma 4.11 we obtain  $\mathcal{H}_{\omega^{\Omega+a}+1} \vdash_1^{\psi(\omega^{\Omega+a})} I_\varphi^{<\Omega}(n)$ , which in turn yields  $\mathcal{H}_{\omega^{\Omega+a}+1} \vdash_1^{\psi(\omega^{\Omega+a})} I_\varphi^{<\psi(\omega^{\Omega+a})}(n)$  by Bounding Lemma 4.10. We conclude  $|n|_\varphi < \psi(\omega^{\Omega+a}) < \psi(\omega^{\Omega \cdot \omega_{1+k}}) = \psi(\Omega^{\omega_{1+k}}) = \vartheta(\Omega \cdot \omega_{1+k})$ .

Second consider Theorem 2.3.3. Let  $Th(\mathbb{N}) + \Pi_1(P)\text{-ID}(\text{Acc}) \vdash R_\varphi(n)$  for an **Acc**-operator  $\varphi(X, x)$ . By Embedding Lemma 4.8 we have  $\mathcal{H}_2 \vdash_3^a I_\varphi^{<\Omega}(n)$  for  $0 < a < \Omega \cdot \omega$ . Lemma 4.9 yields  $\mathcal{H}_2 \vdash_2^a I_\varphi^{<\Omega}(n)$ . Collapsing Lemma 4.11 together with Bounding Lemma 4.10 yields  $\mathcal{H}_{\omega^{\Omega+\omega^a}+1} \vdash_1^{\psi(\omega^{\Omega+\omega^a})} I_\varphi^{<\psi(\omega^{\Omega+\omega^a})}(n)$ . We conclude  $|n|_\varphi < \psi(\omega^{\Omega+\omega^a}) < \psi(\omega^{\omega^{\Omega \cdot \omega}}) = \psi(\Omega^{\omega^\omega}) = \vartheta(\Omega^\omega)$ .

## References

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